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Discrete Mathematics 307 (2007) 764–768

DISCRETE MATHEMATICS

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Note

The Erdős–Pósa property for vertex- and edge-disjoint odd cycles in graphs on orientable surfaces

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Received 14 March 2006; received in revised form 9 June 2006; accepted 10 July 2006

Available online 24 August 2006

Abstract

We prove that for any orientable surface S and any non-negative integer k, there exists an integer $f_S(k)$ such that every graph G embeddable in S has either k vertex-disjoint odd cycles or a vertex set A of cardinality at most $f_S(k)$ such that $G - A$ is bipartite. Such a property is called the Erdős–Pósa property for odd cycles. We also show its edge version. As Reed [Mangoes and blueberries, Combinatorica 19 (1999) 267–296] pointed out, the Erdős–Pósa property for odd cycles do not hold for all non-orientable surfaces.

Keywords: Erdős–Pósa property; Odd cycles; Orientable surfaces

1. Introduction

A family $\mathscr{F}$ of graphs is said to have the *Erdős–Pósa property* if for every integer k, there is an integer $f(k, \mathscr{F})$ such that every graph G contains either k vertex-disjoint subgraphs each isomorphic to a graph in $\mathscr{F}$ or a set C of at most $f(k, \mathscr{F})$ vertices such that $G - C$ has no subgraph isomorphic to a graph in $\mathscr{F}$. The edge version can be considered as well. The term "*Erdős–Pósa property*" arose from [2], in which Erdős and Pósa proved that the family of cycles has this property. Robertson and Seymour [10] extended this to the class of graphs having any fixed planar graph as a minor. Thomassen [12] proved that the family of cycles of length 0 modulo m satisfies the Erdős–Pósa property.

On the other hand, for odd cycles, the situation is different. Lovász characterizes the graphs having no two disjoint odd cycles, using Seymour's result on regular matroids. No such characterization is known for more than two odd cycles. Though the Erdős–Pósa property for odd cycles does not hold in general, Reed [9] pointed out that there exists a cubic projective planar graph which does not contain two edge-disjoint odd cycles, but there is neither a vertex set A nor an edge set B of a bounded cardinality such that $G - A$ and $G - B$ are bipartite. Hence this example shows that the Erdős–Pósa property does not necessarily hold for cycles of length $\not\equiv 0$ modulo m for some m [12].

While the Erdős–Pósa property does not hold for vertex- and edge-disjoint odd cycles in general, these are known to hold for some classes of graphs [5,4,8,13]. Moreover, Reed proved that the Erdős–Pósa property holds for vertex-disjoint odd cycles in planar graphs [9]. For a set A, let $|A|$ denote the cardinality of A.

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doi:10.1016/j.disc.2006.07.008

In this paper, we prove that the Erdős–Pósa property holds for vertex- and edge-disjoint odd cycles in graphs embeddable in an orientable surface, as follows.

Theorem 1. *For any two non-negative integers g and k, there exists an integer $f_g(k)$ such that every graph G embeddable on the orientable surface of genus g has either k vertex-disjoint odd cycles or a vertex set A with $|A| \leqslant f_g(k)$ such that $G - A$ is bipartite.*

Theorem 2. *For any two non-negative integers g and k, there exists an integer $f'_g(k)$ such that every graph G embeddable on the orientable surface of genus g has either k edge-disjoint odd cycles or an edge set B with $|B| \leqslant f'_g(k)$ such that $G - B$ is bipartite.*

Theorems 1 and 2 for the sphere have already been proved by Reed [9] and Berge and Reed [1], respectively. After that, Král' and Voss [6] determined the exact bound for $f'_0(k)$, namely $f'_0(k) = 2k$.

2. Proof of the theorems

Let S_g denote the orientable closed surface of genus g. An *embedding* (or a *map*) on a closed surface F^2 means a fixed embedding of some simple graph on F^2. A face of an embedding G is said to be *even* (resp., *odd*) if its facial closed walk has even (resp., odd) length. An embedding G is said to be *even* if each face of G is even. Note that every even embedding on the sphere is bipartite, but this does not hold for all non-spherical surfaces.

Let F^2 be a non-spherical surface and let ℓ be a simple closed curve on F^2. We say that ℓ is *essential* if ℓ does not bound a 2-cell on F^2, and that ℓ is *separating* if the surface $F^2 - \ell$ is disconnected. Clearly, if ℓ is non-separating, then ℓ is essential. A cycle C of an embedding G on F^2 is said to be *essential* (resp., *separating*) if C is essential (resp., separating) as a simple closed curve on F^2.

The *face-width* (or *representativity*) of an embedding G on a non-spherical surface F^2, denoted $\mathrm{fw}(G)$, is the minimum number of intersecting points of G and ℓ, where ℓ ranges over all essential closed curves on F^2. Note that the face-width for a plane graph cannot be defined since the plane (or the sphere) admits no essential closed curve. For the notation concerning graphs on surfaces, the readers should refer to [7].

Lemma 3. *For any two positive integers k and g, there exists an integer $N_g(k)$ such that every embedding G on the orientable surface S_g with $\mathrm{fw}(G) \geqslant N_g(k)$ has k disjoint homotopic non-separating cycles. In particular, if G is a non-bipartite even embedding on S_g, then these cycles can be taken to have odd length.*

Proof. We prove only the latter, since our proof also works for the former. Let G be an even embedding on S_g. We first observe that any two homotopic cycles of G have the same parity of length. Second, there is a finite set Ω of pairwise non-homotopic essential non-separating simple closed curves on S_g such that for any even embedding G on S_g, there is a closed walk of odd length of G homotopic to some element of Ω. Third, for any embedding H on S_g, there is an integer N_H such that any embedding on S_g with face-width at least N_H has H as a surface minor [10]. Using the above three facts, we have only to take an embedding H on S_g with k disjoint homotopic cycles for each element of Ω and put $N_g(k) = N_H$. □

Let G be a graph and let A be a vertex set or an edge set of G. We say that A is *bipartizing* if $G - A$ is bipartite.

Proof of Theorem 1. We shall define a function $f_g(k)$ such that every embedding G on S_g has either k disjoint odd cycles or a bipartizing vertex set A with $|A| \leqslant f_g(k)$.

We use induction on g. By the result in [9], the value $f_0(k)$ exists, and hence we get the first step of induction when $g = 0$. Therefore, we assume that $f_{g'}(k)$ exists for any $g' < g$ and consider the case when the genus is exactly $g \geqslant 1$.

Let

$$h = \max\{N_g(2f_{g-1}(k) + 2f_0(k) + 3),\ N_g(k) + f_{g-1}(k) + f_0(k)\},$$

where $N_g(k)$ is the number in Lemma 3. Note that h depends only on k and g since so are $N_g(k)$ and $f_{g'}(k)$ with $g' < g$.

Let

$$f_g(k) = \max\left\{ \max_{\substack{g_1, g_2 > 0 \\ g_1+g_2=g}} \{h - 1 + f_{g_1}(k) + f_{g_2}(k)\}, h - 1 + f_{g-1}(k) \right\}.$$

Case 1: G admits an essential simple closed curve ℓ intersecting G at most $h-1$ times.

We may assume that ℓ intersects G only at vertices. Then we can take the vertex set S of G intersected by ℓ such that $|S| \leqslant h-1$. We first suppose that ℓ separates S_g. Then S_g is separated into two punctured orientable surfaces. Pasting a disk to each boundary component, we obtain two non-spherical closed orientable surfaces S_{g_1} and S_{g_2}, where $g_1, g_2 > 0$ and $g_1 + g_2 = g$. Let G_i be the component of $G - S$ on S_{g_i}, for $i = 1, 2$. By the induction hypothesis, each G_i has either k disjoint odd cycles, or a bipartizing vertex set of cardinality at most $f_{g_i}(k)$. Therefore, G has either k disjoint odd cycles or a bipartizing vertex set of cardinality at most $h - 1 + f_{g_1}(k) + f_{g_2}(k) \leqslant f_g(k)$. Secondly, we suppose that ℓ is non-separating. Similarly to the former case, $G - S$ is an embedding on S_{g-1}. By the induction hypothesis, $G - S$ has either k disjoint odd cycles or a bipartizing vertex set of cardinality at most $f_{g-1}(k)$. Hence G has either k disjoint odd cycles or a bipartizing vertex set of cardinality at most $h - 1 + f_{g-1}(k) \leqslant f_g(k)$.

Case 2: The face-width of G is at least h.

Put $l = f_{g-1}(k) + f_0(k) + 1$. By the definition of h and Lemma 3, G has $2l + 1$ disjoint homotopic essential non-separating cycles $C_1, \ldots, C_{2l+1}$ in this order. Let $H = G - V(C_{l+1})$, which is embedded in S_{g-1}. By the induction hypothesis, H has either k disjoint odd cycles or a bipartizing vertex set A with $|A| \leqslant f_{g-1}(k)$. In the former case, the k disjoint odd cycles in H are required ones in G, and hence we consider the latter.

Next, let us consider the annular subgraph Q of G bounded by C_1 and C_{2l+1}. If Q contains k disjoint odd cycles, then we are done. Hence we may assume that Q has a bipartizing vertex set A' with $|A'| \leqslant f_0(k)$, since Q is planar.

Let $G' = G - A - A'$. We claim that G' has no odd face. Suppose it has. Since there are no odd faces in Q and H, the odd face must contain both a vertex of C_{l+1} and a vertex outside Q. But this is impossible since we deleted at most $f_0(k) + f_{g-1}(k) < l$ vertices in the annulus Q bounded by C_1 and C_{2l+1} and any curve from a vertex of C_{l+1} to a vertex outside the annulus intersects at least l vertices in G. Hence there are no odd faces in G'.

If G' is bipartite, then $A \cup A'$ is a required bipartizing vertex set of G, since

$$|A \cup A'| \leqslant |A| + |A'| \leqslant f_{g-1}(k) + f_0(k) \leqslant f_g(k).$$

Hence we suppose that G' is non-bipartite. Since G' is obtained from G by removing at most $f_{g-1}(k) + f_0(k)$ vertices, the face-width of G' is still at least $h - (f_{g-1}(k) + f_0(k)) \geqslant N_g(k)$. By Lemma 3, G' has k disjoint odd cycles. $\square$

Proof of Theorem 2. Observe that the above proof works for the edge-disjoint case if the face-width is large enough, since we can apply Lemma 3 and find many disjoint homotopic essential cycles. So we consider the case when the face-width is small.

The *dual-width* of G on S_g, denoted $\mathrm{dw}(G)$, is the minimum number of intersecting points of G and ℓ, where ℓ ranges over all essential closed curves intersecting G only at inner points of edges. (The dual-width of G is the length of a shortest essential cycle of the surface dual of G.) If G has small face-width and small dual-width, then we can apply induction with respect to g by removing a few edges, similarly to the proof of Theorem 1. However, G can have small face-width but an arbitrarily large dual-width. We shall handle only this case.

Let v be a vertex of G and let $e_1, \ldots, e_m$ be the edges of G incident to v in this cyclic order (then $\deg_G(v) = m$). For any edge e_i, starting at the vertex v, put $\lfloor m/2 \rfloor$ vertices $v_i^1, \ldots, v_i^{\lfloor m/2 \rfloor}$ in this order. (Each edge of the resulting embedding on the path between v and $v_i^{\lfloor m/2 \rfloor}$ is called an *auxiliary* edge.) Next, join v_i^j and v_{i+1}^j by a path of length 2 for each i and j (where the indices i and $i+1$ are taken modulo m). We call this operation a *patch extension* with respect to v. (See Fig. 1, for example.) Clearly, the *patch* (i.e., the plane graph with outer cycle through $v_1^{\lfloor m/2 \rfloor}, \ldots, v_m^{\lfloor m/2 \rfloor}$) is bipartite. Each face in a patch is called an *auxiliary face* of the resulting embedding. The $\lfloor m/2 \rfloor$ cycles of length $2m$ surrounding v are called the *nested cycles* for v.

Let $\tilde{G}$ be the embedding on S_g obtained from G by the patch extensions with respect to all vertices of G. Note that each edge of $\tilde{G}$ not on any nested cycle corresponds to some edge of G. A non-auxiliary edge and face of $\tilde{G}$ are said to be *intrinsic*. Clearly, the intrinsic edges of $\tilde{G}$ and the edges of G have the one-to-one correspondence, and hence so do

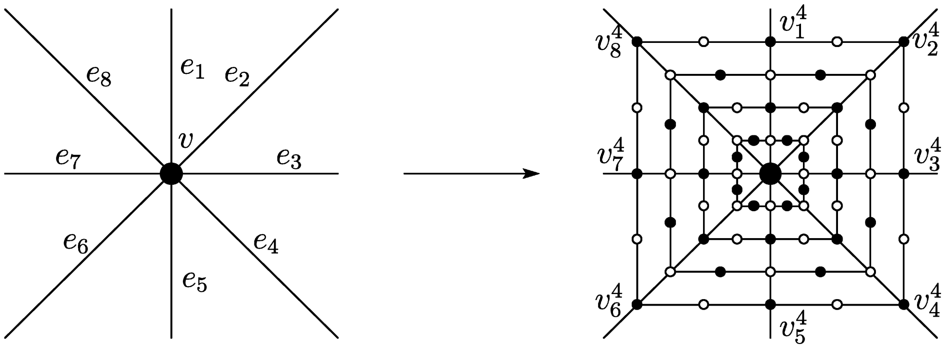

Fig. 1. A patch extension with respect to v.

the intrinsic faces of $\tilde{G}$ and the faces of G. Moreover, every auxiliary face is even, and G and $\tilde{G}$ have the same number of odd faces.

Observe that the face-width of $\tilde{G}$ is greater than or equal to the dual-width of G. Since the dual-width of G is assumed to be large enough, so is the face-width of $\tilde{G}$. Therefore, $\tilde{G}$ satisfies the theorem, as described in the beginning of the proof. That is, $\tilde{G}$ has either k edge-disjoint odd cycles $\tilde{C}_1, \ldots, \tilde{C}_k$ or a bipartizing edge set $\tilde{S}$ of bounded cardinality. In order to complete the proof, we shall prove that G has k edge-disjoint odd cycles corresponding to $\tilde{C}_1, \ldots, \tilde{C}_k$, or a bipartizing edge set S with $|S| \leqslant |\tilde{S}|$.

We first consider the former case. It is an important observation that every odd cycle of $\tilde{G}$ must use an odd number of intrinsic edges of $\tilde{G}$, since the graph in each patch is bipartite and since $v_i^{\lfloor m/2 \rfloor}$ and $v_{i+1}^{\lfloor m/2 \rfloor}$ in the same patch belong to the same partite set of its bipartition, for any i. Hence the cycle, say C_i, in G corresponding to $\tilde{C}_i$ has odd length. Since $\tilde{C}_1, \ldots, \tilde{C}_k$ are edge-disjoint in $\tilde{G}$, so are $C_1, \ldots, C_k$ in G.

Now we consider the latter. Let S be the set of edges of G corresponding to the edges of $\tilde{S}$ which are not on nested cycles. Then we have $|S| \leqslant |\tilde{S}|$.

We claim that $G-S$ is bipartite. If not, then $G-S$ has an odd cycle, say $C = u_0, e_1, u_1, e_2, \ldots, u_{2l}, e_{2l+1}, u_{2l+1} (= u_0)$, where $u_i \in V(G)$ and $e_i \in E(G)$ for each i. For each i, let L_i be the path in $\tilde{G}$ corresponding to e_i. By the definition of S, no edge is deleted from L_i in $\tilde{G} - \tilde{S}$. Hence, $\tilde{G} - \tilde{S}$ has the cycle $\tilde{C} = \bigcup_{i=1}^{2l+1} L_i$ corresponding to C. Since each L_i consists of one intrinsic edge and $\lfloor \deg_G(u_{i-1})/2 \rfloor + \lfloor \deg_G(u_i)/2 \rfloor$ auxiliary edges in $\tilde{G}$,

$$|\tilde{C}| = \sum_{i=1}^{2l+1} |L_i| = 2l + 1 + \sum_{i=1}^{2l+1} 2 \left\lfloor \frac{\deg_G(v_i)}{2} \right\rfloor \equiv 1 \ (\mathrm{mod}\ 2).$$

This contradicts that $\tilde{G} - \tilde{S}$ is bipartite. $\square$

As is mentioned in Section 1, it has been proved by Král' and Voss [6] that $f_0'(k) = 2k$. Let us estimate $f_1'(k)$ for the torus S_1, using the result of de Graaf and Schrijver [3], i.e., every toroidal embedding G with $\mathrm{fw}(G) \geqslant \frac{3}{2}r$ has a toroidal grid $C_r \times C_r$ as a surface minor, and hence G has r disjoint homotopic essential cycles.

Proposition 4. $f_1'(k) \leqslant 14k + 4$.

Proof. Let G be any embedding on the torus S_1. We may suppose that $\mathrm{fw}(G) \geqslant \mathrm{dw}(G)$, as in the proof of Theorem 2. Suppose that $\mathrm{dw}(G) \geqslant 12k + 5$. (For otherwise, removing at most $12k + 4$ edges, we obtain a plane graph, which has a bipartizing edge set of cardinality $2k$, by Král' and Voss's result [6]. Therefore, G has a bipartizing edge set of cardinality $\leqslant 12k + 4 + 2k = 14k + 4$.)

Since $\mathrm{fw}(G) \geqslant 12k + 5$, G has $8k + 3$ disjoint homotopic essential cycles $C_1, \ldots, C_{8k+3}$ by the above mentioned de Graaf and Schrijver's result. Then, considering two annular subgraphs $H = G - V(C_{4k+2})$ and Q bounded by C_1 and C_{8k+3} and removing at most $2k + 2k = 4k$ edges from G, we get an even embedding G' from G. (If G' has an odd face f, then f must have a vertex of C_{4k+2} and a vertex not contained in Q. However, this is impossible, since we deleted at most $4k$ edges. The details should be referred to the proof of Theorem 2.) If G' is bipartite, then the edge removed is a required bipartizing edge set of cardinality at most $4k < 14k + 5$.

Hence we suppose that G' is non-bipartite. Since $\mathrm{fw}(G') \geqslant 12k + 5 - 4k = 8k + 5$, G' has $C_{5k+3} \times C_{5k+3}$ as a surface minor, by the above result. Let $A_1, \ldots, A_{5k+3}$ be $5k + 3$ disjoint homotopic essential cycles in G' corresponding to those in $C_{5k+3} \times C_{5k+3}$, Note that $A_1, \ldots, A_{5k+3}$ have the same parity of length, since they are homotopic and since G' is an even embedding. Moreover, G has another set of disjoint homotopic essential cycles orthogonal to A_i's, since G has $C_{5k+3} \times C_{5k+3}$ as a surface minor. Let $B_1, \ldots, B_{5k+3}$ be such cycles, which have the same parity of length. Observe that A_1 or B_1 have odd length, since A_1 and B_1 cut the torus into a disk and since G' is non-bipartite. Hence G has $5k + 3$ $(\geqslant k)$ odd cycles. $\square$

In order to get a linear bound for $f'_g(k)$ with $g \geqslant 2$, it suffices to prove that the face-width bounded by a linear function of k guarantees the existence of k disjoint homotopic cycles with a specified homotopy type on S_g, as in the above proof. However, it does not seem to be easy.

Acknowledgements

We are grateful to the anonymous two referees to give us helpful comments pointing out our small mistakes and suggestions improving our expressions.

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